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Note

On the Number of 8×8 Latin Squares

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The numbers of non-isomorphic Latin squares of order 8 under the action of various different symmetry groups have been reported by Wells in 1967, by Brown in 1968 and by Arlazarov and coworkers in 1978. The result of Wells has also been independently verified by Bammel and Rothstein in 1975. As an intermediate step towards finding a complete list of projective planes of order 9, we have to generate all the Latin squares of order 8. We find numbers that agree with Wells, but different from those of Brown and Arlazarov and coworkers. These new numbers are reported in this note. We also give the distributions of these numbers according to the size of the automorphism group, which allows one to easily cross check the results.

1. INTRODUCTION

A *Latin square* of order n is an $n \times n$ matrix satisfying the following properties:

1. all the entries are integers between 1 and n,
2. in every row, no entry is repeated, and
3. in every column, no entry is repeated.

The book by Denes and Keedwell [4] is an excellent reference to the subject of Latin squares and we shall follow their terminology. A Latin square is said to be *reduced* if in the first row and column, its elements $1, 2, \ldots, n$ occur in natural order. Two Latin squares are said to be *isotopic* if one can be transformed into the other by rearranging rows, rearranging

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columns and renaming elements. The isotopic relation divides the Latin squares into equivalence classes, called *isotopy classes*. A Latin square can also be transformed into others, called its *conjugates*, by the *conjugate operations*, which are generated by the operations of *transposition* and *row inverse*. The row inverse operation is defined by treating each row of the Latin square as a permutation in image form and by replacing it with its inverse. The combinations of the conjugate operations generate a group isomorphic to S_3, the symmetric group on three letters. A set of Latin squares which comprises all the members of an isotopy class together with their conjugates is called a *main class*.

There are several published results regarding the number of Latin squares of order 8. In 1967, Wells [9] gave the number of reduced Latin squares as 535,281,401,856. In 1968, Brown [3] gave the number of isotopy classes as 1,676,257. In 1978, Arlzarov *et al.* [1] gave the number of main classes as 283,640. They also mentioned that their numbers can be checked against those of Wells. However, the two sets of numbers are not consistent with one another! Arlazarov's numbers give fewer reduced Latin squares. In 1975, while counting the number of reduced Latin squares of order 9, Bammel and Rothstein [2] verified the correctness of the results of Wells. So, it seems that Arlazarov has missed some squares. Unfortunately, Brown did not give enough details to check his numbers against the others.

As an intermediate step towards deciding whether there can be any new projective planes of order 9, we have to generate one representative from each main class of Latin squares of order 8. We found 283,657 main classes, which is 17 more than Arlazarov *et al.* had found. We also found 1,676,267 isotopy classes, which is 10 more than Brown had found. Our numbers are consistent with one another and also consistent with those of Wells.

In Section 2, we shall present the detailed results and indicate how one can convert one set of numbers to another. Section 3 contains a very brief description of our method and in Section 4, we make some comments about the validity of computer generated results, especially in enumerative searches.

2. The Results and Their Verification

MacMahon in [6] pointed out that the number of Latin squares of order n, U_n, and the number of reduced Latin squares of order n, T_n, is related by $U_n = n!(n-1)!T_n$. On the other hand, under the action of a symmetry group G, the set of Latin squares is divided into orbits or classes. The size of each orbit can be determined if one knows the size of the

TABLE I

Isotopy Class Distribution by Automorphism Group Size

|Auto|	Number	|Auto|	Number	|Auto|	Number
1	1,644,434	10	12	96	8
2	28,767	12	55	128	12
3	310	16	172	192	6
4	1854	24	21	256	3
5	12	32	36	512	2
6	136	42	5	1536	1
7	2	48	7	10,752	1
8	397	64	14		

automorphism group of any of the Latin squares in the orbit. Let A be a Latin square, A^G be its orbit, and G_A be its automorphism group. Then we have $|A^G| = |G|/|G_A|$. When applied to Latin squares, we have

$$U_n = \sum_{\text{one } A \text{ from each orbit}} \frac{|G|}{|G_A|}.$$

If one knows the distribution of the classes according to the size of their automorphism groups, then the formula can be restated as

$$U_n = |G| \sum_i \frac{\text{number of classes with } |G_A| = i}{i}.$$

For isotopy classes, $|G| = (n!)^3$. The distribution of the isotopy classes for

TABLE II

Main Class Distribution by Automorphism Group Size

|Auto|	Number	|Auto|	Number	|Auto|	Number
1	270,611	18	9	126	1
2	11,119	20	2	128	4
3	213	21	1	192	5
4	1089	24	28	256	4
5	1	32	34	288	1
6	158	36	2	384	4
8	227	48	8	576	2
9	1	64	11	1536	3
10	3	72	1	3072	2
12	35	84	1	9216	1
16	69	96	6	64,512	1

TABLE III

Differences with Arlazarov's Numbers

|Auto|	Arlazarov	New
1	270,601	270,611
2	11,115	11,119
3	212	213
4	1088	1089
6	157	158

Latin squares of order 8 according to their automorphism group sizes is given in Table I. Now, one can compute T_8 by

$$T_8 = 8(8!) \sum_i \frac{\text{number of isotopy classes with } |G_A| = i}{i}.$$

For the main classes, $|G| = 3!(n!)^3$. The distribution of the main classes for Latin squares of order 8 according to their automorphism group sizes is given in Table II. Now, one can compute T_8 by

$$T_8 = 48(8!) \sum_i \frac{\text{number of main classes with } |G_A| = i}{i}.$$

The reader is invited to verify that in both cases, we obtain the value of T_8 as found by Wells. Table III shows where our numbers differ from those of Arlazarov. In every instance, we found more squares.

In order to convert from main classes to isotopy classes, one has to know the distribution of the number of conjugate isotopy classes generated by each main class. The distribution of main classes by the number of conjugates is given in Table IV.

Brown has also classified the isotopy classes by the number of even and odd rows when a row is treated as a permutation of the first row. By

TABLE IV

Main Class Distribution by Number of Conjugates

No. of Conjugates	No. of Squares
1	175
2	191
3	8012
6	275,279

TABLE V

Isotopy Class Distribution
by Number of Even and Odd Rows

(even, odd)	Brown	New
(4,4)	466,867	466,867
(5,3)	720,840	720,840
(6,2)	372,344	372,350
(7,1)	101,782	101,786
(8,0)	14,424	14,424
Total	1,676,257	1,676,267

choosing a different first row, if necessary, the number of even rows is at least equal to the number of odd rows. Table V gives a summary of our results as compared to his. Again, where there are differences, we find more squares.

3. A Brief Description of Our Method

Our method of generating the squares is based on Brown's idea of extending the square one row at a time while restricting the first four rows to be even ones. Isomorph rejection is performed for each completed $r \times 8$ rectangle, for r from 2 to 5, and then at 8. We also use the row inverse operation to cut the number of cases. When we turn off the pruning by row inverse, we get Brown's counts of the numbers of non-isomorphic $r \times 8$ rectangles, for r from 2 to 4. After we have a completed 8×8, we apply the conjugate operations in order to reject squares which are not representatives for the main classes.

After we found the discrepancy between our results and those of Brown and Arlazarov, we modified the program to perform a limited version of consistency checking based on the idea in [5]. For each main class, we keep track of the expected number of conjugates that have not yet been encountered in the enumeration. This value is initialized based on the number of conjugation operations fixing the square. When a square is rejected because of a conjugate operation, we determine the exact square that it is equivalent to and reduce its count. This program runs about three times slower, but we found that the enumeration is internally consistent.

4. Conclusion

The history of the number of Latin squares of order 8 seems to parallel that of order 7, where Sade found some missing squares in Norton's list

[7, 8]. It points out the difficulty of performing an accurate enumeration. With the increasing use of computers in mathematics, the correctness of such "proofs" is very difficult to determine. We should borrow an idea from the physical sciences, where a new result is accepted only after it has been independently verified. The published results should contain enough information to allow independent cross checking of values besides the final numbers. Internal consistency checking should also be used as much as possible, even when it is expensive.

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